Soil Mechanics I
6 – Consolidation

1 Definition
2 Theory of one-dimensional consolidation
3 Application: Settlement
4 Coefficient of consolidation; lab determination
Undrained loading $\rightarrow$ excess pore pressure $\Delta u$

$$u = u_0 + \Delta u$$

$\rightarrow$ hydraulic gradient

$\rightarrow$ seepage

$\rightarrow$ volume changes

$\rightarrow$ a change in effective stress $\Delta \sigma'$

With time decrease in $\Delta u$

$$u = u_0 = \text{end of consolidation}$$

NB: during consolidation the rate of deformation is not linear (which will be shown later in the theory)
1 Consolidation ≡ dissipation of excess pore pressure

2 Consolidation is a coupled process: seepage + volume change

\((\varepsilon_V \neq 0; \varepsilon_V >0, <0)\)
Generally: consolidation in 3D

Special case: large loaded area + relatively thin soil layer → 1D

In-situ:

Lab model:
Spring Analogy:

Pressures \((\sigma, u)\) at different given times \(t_1, t_2\ldots\): isochrones
One-dimensional consolidation

Spring Analogy (a more realistic one + both-side drainage):

\[ \sigma'_{vo} + \Delta \sigma \]

\[ \Delta \sigma \rightarrow \Delta u \]

\[ \sigma'_{vo} \quad \Delta u \]

\[ \Delta \sigma'_{t1} \]

\[ \Delta u_{t1} \]

\[ t = 0 \]

\[ t_1 \quad t_2 \]

\[ t_n \]
Degree of consolidation $U_z$

\[ (= \frac{\Delta V_z(t)}{\Delta V_{z,\text{fin}}}) \]

\[ \sigma' - \sigma'_1 = (\sigma'_2 - \sigma'_1) - u = \Delta u - u = u_i - u \]

\[ U_z = (e - e_i) / (e_2 - e_i) = (\sigma' - \sigma'_1) / (\sigma'_2 - \sigma'_1) = (u_i - u) / u_i = 1 - u / u_i \]
Average degree of consolidation $U$ ($U_{avg}$) for both-side drainage

$= \frac{\Delta V(t)}{\Delta V_{fin}}$

One-dimensional consolidation: $U$ directly linked to settlement
Assumptions:

1. Saturated soil (S=1)

2. Water and grains incompressible

   1 + 2: principle of effective stress valid

3. Darcy's law ($v = k i$, $k$ = const.)

4. Homogeneous, linearly elastic soils, small deformations

5. Compression + seepage in 1-D

   (+ top and bottom drainage)
Continuity of flow ($Q_{out} = Q_{in}$) + **compressibility of the soil ($\varepsilon_y \neq 0$)**
Hydraulic gradient in depth $z$:

$$i_z = \partial(\frac{u}{\gamma_w}) / \partial z = \frac{1}{\gamma_w} \partial u / \partial z$$

Hydraulic gradient in depth $z+dz$:

$$i_{z+dz} = \frac{1}{\gamma_w} \partial u / \partial z + \frac{1}{\gamma_w} \partial^2 u / \partial z^2 \, dz$$

Darcy:

$$dQ = k \, i \, dx \, dy \, dt$$

$$dQ_{out} = k \frac{1}{\gamma_w} \partial u / \partial z \, dx \, dy \, dt$$

$$dQ_{in} = k \frac{1}{\gamma_w} (\partial u / \partial z + \partial^2 u / \partial z^2 \, dz) \, dx \, dy \, dt$$

Volume change:

$$dQ_{out} - dQ_{in} = - k / \gamma_w \, \partial^2 u / \partial z^2 \, dz \, dx \, dy \, dt$$

$$(dx \, dy = 1: \, dQ_{out} - dQ_{in} = - k / \gamma_w \, \partial^2 u / \partial z^2 \, dz \, dt)$$
Volume change - compressibility:

Using coefficient of compressibility (e.g.):

\[ a_v = \frac{-\Delta e}{\sigma'} = -\frac{de}{d\sigma'} \]

Volume change = settlement

\[ \Delta dz = -\frac{de}{1+e_1} dz = a_v \frac{d\sigma'}{(1+e_1)} dz = \]

\[ = m_v d\sigma' dz = \frac{1}{E_{oed}} d\sigma' dz \]

constant total stress (\(\Delta \sigma = 0\)):

\[ \Delta \sigma' = -\Delta u \]

\[ \Delta dz = -\frac{1}{E_{oed}} du dz \]

\[ du = \frac{\partial u}{\partial t} dt \]

\[ \Delta dz = -\frac{1}{E_{oed}} \frac{\partial u}{\partial t} dt dz \]
One-dimensional consolidation – Terzaghi’s Theory

\[ \Delta \sigma = \sigma'_2 - \sigma'_1 - u_i \]

\[ dQ_{out} - dQ_{in} = - \frac{k}{\gamma_w} \frac{\partial^2 u}{\partial z^2} \, dz \, dt \]

\[ \Delta dz = - \frac{1}{E_{oed}} \frac{\partial u}{\partial t} \, dt \, dz \]

\[ \frac{\partial u}{\partial t} = k \frac{E_{oed}}{\gamma_w} \frac{\partial^2 u}{\partial z^2} \]

\[ k \frac{E_{oed}}{\gamma_w} = c_v \; \text{... coefficient of consolidation} \]

\[ \frac{\partial u}{\partial t} = c_v \frac{\partial^2 u}{\partial z^2} \]
\[ \partial u / \partial t = c_v \partial^2 u / \partial z^2 \]

Mathematical solution – initial and boundary conditions

\[
t = 0: \quad \Delta u = u_1 = \Delta \sigma = \Delta \sigma' = \sigma_2' - \sigma_1' \\
z = 0 ; \ z = 2H: \ \Delta u = 0
\]

the initial excess pore pressure is a function of depth \( z \):

\[
u = \sum_{n=1}^{\infty} \left( \frac{1}{H} \int_{z=0}^{z=2H} u_i \sin \frac{n\pi z}{2H} \, dz \right) \sin \frac{n\pi z}{2H} \exp \left( -\frac{c_v n^2 \pi^2}{4H^2} \right)
\]

the initial excess pore pressure varies linearly with depth \( z \) (or constant):

\[
u = (\sigma_2' - \sigma_1') \sum_{n=0}^{\infty} \frac{4}{(2n + 1)\pi} \sin \left( \frac{2n + 1}{2} \frac{\pi}{H} \frac{z}{H} \right) \times \exp \left\{ \frac{(2n + 1)^2}{4} k(1 + e_1) \frac{t}{H^2} \right\}
\]

\[
Z = \frac{z}{H} \text{ (dimensionless depth)} \\
T = c_v \frac{t}{H^2} \text{ (dimensionless time = “time factor“)}
\]
Degree of consolidation

\[ U_z = 1 - \frac{u}{u_1} = 1 - \sum (f_1(Z) \times f_2(T)) \]
A clay layer of \( h = 6 \text{m} \) and \( c_v = 6 \times 10^{-8} \text{m}^2\text{s}^{-1} \) is drained from the top and bottom. What is the degree of consolidation at 1/20, 1/4, 1/2, 3/4 of the clay thickness in one year after an undrained loading.

\[
T = \frac{c_v t}{H^2} = \frac{6 \times 10^{-8} \times 365 \times 24 \times 3600}{9} = 0.21 \approx 0.2
\]

(h = 2H)

Depth ½ h: \( Z = z / H = H / H = 1 \) \( U_z \approx 0.25 = 25\% \)

Depth ¼ h: \( Z = z / H = (\frac{1}{4} 2H) / H = 0.5 \) \( U_z \approx 0.45 = 45\% \)

Depth ¾ h: \( Z = z / H = (\frac{3}{4} 2H) / H = 1.5 \) \( U_z \approx 0.45 = 45\% \)

Depth 1/20 h: \( Z = z / H = (1/20 2H) / H = 0.1 \) \( U_z \approx 0.9 = 90\% \)
U % = 100 (1 - ∑f(T))

Graphically (depending on boundary and initial conditions...)

For U < 60%   T ≈ π/4 U^2
Average Degree of Consolidation $U$ ($U_{avg}$)

$$U \% = 100 \left(1 - \sum f(T)\right)$$

graphically (for $u$ constant or linear with depth)
A clay layer of \( h = 6 \text{ m} \) and \( c_v = 6 \times 10^{-8} \text{ m}^2\text{s}^{-1} \) is drained from the top and bottom. What is the degree of consolidation at \( 1/20, 1/4, 1/2, 3/4 \) of the clay thickness in one year after an undrained loading.

a) What part of the final settlement takes place in one year after loading?
b) When does 95% of the final settlement happen?

\[
a) T = 0.2 \rightarrow U = 50% \\
U = \frac{s(t)}{s_{\text{final}}} \\
1 \text{ year: } 50\% \text{ of final settlement}
\]

\[
b) T = \frac{c_v \ t}{H^2} \\
t = \frac{T \ H^2}{c_v} \\
U = 0.95 \rightarrow T \approx 1.2 \\
U = 0.95 \rightarrow t \approx 1.2 \times 3^2 / (6 \times 10^{-8}) = 1.8 \times 10^8 \text{ s} \approx 6 \text{ years}
\]
Coefficient of Consolidation $c_v$ - determination

Oedometer: log t method (Casagrande)
Coefficient of Consolidation $c_v$ - determination

Oedometer: log t method (Casagrande)

...definition of creep by EOP (End Of Primary) compression
Oedometer: $\sqrt{t}$ method (Taylor)
On the theoretical curve $U$ vs $\sqrt{T}$ (right) Taylor noted, that the initial part of the relationship is linear, and that after decreasing its slope by 15%, the intersection corresponds to $T$ for $U=0.9$. It is used in curve fitting method (left)
Coefficient of Consolidation $c_v$ - determination

Typical values of $c_v$

**Clay**

$n \times 10^{-8} \text{ m}^2\text{s}^{-1}$

**Silt**

$n \times 10^{-4} \text{ m}^2\text{s}^{-1}$

$(\pm$ kaolin in Lab Class No4)

$1\times10^{-8} \text{ m}^2\text{s}^{-1} \approx 0.3 \text{ m}^2/\text{year}$
http://labmz1.natur.cuni.cz/~bhc/s/sm1/


Further reading: